

Green's function for a n-dimensional closed, static universe and with a spherical boundary

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Abstract

We construct the Hadamard Green's function by using the eigenfunction, which are obtained by solving the wave equation for the massless conformal scalar field on the S^{n-1} of a n-dimensional closed, static universe. We also consider the half space case with both the Dirichlet and the Neumann boundary conditions. Solving of eigenfunction and eigenvalues of the corresponding field equation is interesting since the Casimir energy could be calculated analytically by various methods.

I. INTRODUCTION

In 1948, Casimir [1] showed that one consequence of the zero-point quantized field is an attractive force between two uncharged, perfectly conducting parallel plates. The principal message of this force is that changes in infinite vacuum energy of the quantized field can be finite and observable [2,3].

The Casimir energy leads to negative energy between the plates, depends on the distance between plates and independent plate area, as well as similar calculation were made for sphere [4], for which the Casimir energy was found to be positive which means that the force on the shell is repulsive and depends only on the radius of the shell. The type of Casimir energy values depend on manifolds of different topology and geometry [5]. The

boundary and/or curvature conditions play the important role of the quantized field.

The Casimir effect has also been calculated in curved background spacetime. In 1975, Ford [6] showed that the vacuum energy of a massless conformal scalar field in an Einstein universe by the mode sum method. Ford obtained the renormalized vacuum energy density in this case as

$$\rho = \frac{\hbar c}{480\pi^2 R_0^4}, \quad (1)$$

where, R_0 is the radius of the universe and the pressure is given by $P = \frac{1}{3}\rho$.

Thus the energy momentum tensor is in the same form as that for the classical radiation. In Ford's work on the Casimir effect in an Einstein universe, he showed that the energy is associated with the closed spatial topology. Ford used the mode sum method and employed an exponential cutoff in his calculations. Later, Dowker and Critchley [7] verified Ford's result by using the covariant point-splitting method. Their calculation was not only covariant but also cutoff independent. Even though Dowker and Critchley used the global topology of the universe to construct the Green's function, their result is also expected to be true locally; i.e., the local renormalized energy density and pressure should be given by

$$(\langle T_0^0 \rangle =) \quad \rho = \frac{\hbar c}{480\pi^2 R_0^4}, \quad (-\langle T_1^1 \rangle = -\langle T_2^2 \rangle = -\langle T_3^3 \rangle =) \quad P = \frac{1}{3}\rho. \quad (2)$$

Three often used regularization techniques of the quantum vacuum energy in a curved background spacetimes are the mode sum method [6], the zeta function regularization technique [10] and the point-splitting method. One of them, the most powerful method available is the point-splitting regularization technique [8,9]. This method works with point separated functions. In the Hamiltonian, the scalar field $\Phi(x)$ behaves like a field operator due to the canonical quantization scheme. However, the products of field operators and their covariant derivatives are at the same point which causes the vacuum expectation value of the energy momentum tensor to diverge. To eliminate these divergent quantities, one of the field operators $\Phi(x)$ in each product in the Hamiltonian is replaced by $\Phi(x')$, where x' is some point near x . Thus the separated function $\langle 0 | \Phi(x) \Phi(x') | 0 \rangle$ transforms like the product two

functions, one at x , the other at x' . The two point function $\langle 0 | \Phi(x) \Phi(x') | 0 \rangle$ is called the positive Wightmann function and is denoted by

$$G^{(+)}(x, x') = \langle 0 | \Phi(x) \Phi(x') | 0 \rangle. \quad (3)$$

The vacuum expectation value of the anticommutator of the field is called the Hadamard Green's function, which is defined by [8]

$$G^{(1)}(x, x') = \langle 0 | \{ \Phi(x) \Phi(x') + \Phi(x') \Phi(x) \} | 0 \rangle, \quad (4)$$

where $G^{(1)}(x, x')$ and $G^{(+)}(x, x')$ are related by

$$G^{(1)}(x, x') = 2 \operatorname{Re} G^{(+)}(x, x'). \quad (5)$$

In this work, we use the above physical motivation for the massless conformal scalar field in a n -dimensional closed, static universe. Firstly, we will construct the Hadamard Green's function for the case of a n -dimensional static, closed universe, using the separated function idea. Secondly, we construct the Green's function directly by using the wave function satisfying the boundary conditions. The half space case (as denoted by Kennedy and Unwin [11]) is very important in the sense that despite being in curved spacetime, and with a spherical boundary. Our results confirm the Green's function for the full n -dimensional closed, static universe as an image sum and obtained the necessary Green's function for the n -dimensional half space case by locating an image charge in dual region.

We have organized this paper as follows. In section II. we calculate the mode function for the massless conformal scalar field in a n -dimensional closed, static universe. In section III. we constructed the Hadamard Green's function by using the eigenfunction, which are obtained by solving wave equation for the massless conformal scalar field on S^{n-1} ($n \geq 4$, n is the dimension of the spacetime). In section IV. we study the Green's function for the massless conformal scalar field with a spherical boundary at $\chi_0 = \frac{\pi}{2}$ and with the interior geometry represented by the closed, static n -dimensional universe (this case is also called the half universe). We consider both the Dirichlet and the Neumann boundary conditions.

II. THE MODE FUNCTION

We shall use hyperspherical polar coordinates $(\chi, \theta, \phi) = (\chi, \theta_1, \theta_2, \theta_3, \dots, \theta_\mu, \phi)$; $\mu = 1, 2, 3, \dots, n-3$. Where n is the spacetime dimension. The line element in the time orthogonal form

$$ds^2 = dt^2 - R_0^2 dl^2, \quad (6)$$

where R_0 is the radius of the universe and is a constant. In a static-closed model the part dl^2 follows by imagining that the space sections of fixed world time t are embedded in a n -dimensional space with coordinates $x_1, x_2, x_3, \dots, x_{n-1}, w$ and line element $dl^2 = dx_1^2 + dx_2^2 + \dots + dx_{n-1}^2 + dw^2$. Our $n-1$ dimensional space is the surface of the sphere $x_1^2 + x_2^2 + \dots + x_{n-1}^2 + w^2 = R_0^2$ at fixed R_0 . Spherical coordinates for this space are

$$\begin{aligned} x_1 &= R_0 \sin \chi \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-3} \cos \phi, \\ x_2 &= R_0 \sin \chi \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-3} \sin \phi, \\ x_3 &= R_0 \sin \chi \sin \theta_1 \sin \theta_2 \dots \cos \theta_{n-3}, \\ &\vdots \\ x_{n-1} &= R_0 \sin \chi \cos \theta_1, \\ w &= R_0 \cos \chi. \end{aligned} \quad (7)$$

Where $\chi \in [0, \pi]$, $\theta_\mu \in [0, \pi]$ $\mu = 1, 2, 3, \dots, n-3$, and $\phi \in [0, 2\pi]$.

Thus, the general metric for a static, closed n -dimensional spacetimes has the form

$$\begin{aligned} ds^2 &= dt^2 - R_0^2 [d\chi^2 + \sin^2 \chi d\theta_1^2 + \sin^2 \chi \sin^2 \theta_1 d\theta_2^2 + \dots \\ &\quad + \sin^2 \chi \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{n-4} d\theta_{n-3}^2 \\ &\quad + \sin^2 \chi \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{n-4} \sin^2 \theta_{n-3} d\phi^2]. \end{aligned} \quad (8)$$

We shall consider conformal massless scalar field $\Phi(x)$ on this curved background geometry. It satisfies the field equation

$$\square \Phi(x) + \xi(n) R \Phi(x) = 0. \quad (9)$$

(We follow the sign convention in [8]) Where $\xi(n) = \frac{1}{4} \frac{(n-2)}{(n-1)}$ is the dimensional parameter determining the coupling between $\Phi(x)$ and the Ricci scalar R . The Ricci scalar for the metric Eq.(8) is given by

$$R = \frac{1}{R_0^2} (n-2)(n-1) \quad \text{for } n \geq 2. \quad (10)$$

We solve Eq. (9) using the method of separation of variables. In general, for $n \geq 4$ we can write

$$\begin{aligned} \Phi(x) &= c_{0n} e^{-i\omega t} X(\chi) X_1(\theta_1) X_2(\theta_2) \dots X_{n-4}(\theta_{n-4}) Y_{lm}(\theta_{n-3}, \phi) \\ \text{with } \omega &= \frac{N}{R_0}. \end{aligned} \quad (11)$$

Where Y_{lm} is the spherical harmonic and c_{0n} is appropriate normalization constant, and

$$(N_2 =) l = 0, 1, 2, 3, \dots, \text{ and } m = -l, -l+1, \dots, 0, 1, \dots, l-1, l. \quad (12)$$

We now calculate the mode function for $n \geq 4$. The $X(\chi)$ and $X_\mu(\theta_\mu)$ ($\mu = 1, 2, 3, \dots, n-4$) satisfy the following differential equations

$$\begin{aligned} & \frac{1}{\sin^{n-4} \chi} \frac{d}{d\chi} \left(\sin^{n-2} \chi \frac{d}{d\chi} X(\chi) \right) \\ & + \left[\left(N^2 - \frac{(n-2)^2}{4} \right) \sin^2 \chi - N_{n-2} (N_{n-2} + n - 3) \right] X(\chi) = 0, \text{ and} \quad (13) \\ & \frac{1}{\sin^{n-\mu-4} \theta_\mu} \frac{d}{d\theta_\mu} \left(\sin^{n-\mu-2} \theta_\mu \frac{d}{d\theta_\mu} X_\mu(\theta_\mu) \right) \\ & + [N_{n-\mu-1} (N_{n-\mu-1} + n - \mu - 2) \sin^2 \theta_\mu - N_{n-\mu-2} (N_{n-\mu-2} + n - \mu - 3)] X_\mu(\theta_\mu) = 0 \end{aligned} \quad (14)$$

with $\mu = 1, 2, 3, 4, \dots, n-4$.

Where $(N_2 =)l, N_3, N_4, N_5, \dots, N_{n-3}, N_{n-2}$ and $N = R_0 \omega$ are the separation parameters.

Notice that we still have not calculated the regularity of the solution in our intervals at $\theta_\mu \in [0, \pi]$ where $\mu = 1, 2, 3, \dots, n-4$; hence N_j ($j = 3, 4, 5, \dots, n-2$) and $N = R_0 \omega$ remain continuous parameters while l and m take the values $(N_2 =)l = 0, 1, 2, 3, 4, 5, 6, \dots$, $m = -l, -l+1, \dots, 0, 1, \dots, l-1, l$.

Making the following substitutions for each one in Eqs. (13) and (14)

$$X(\chi) \propto \sin^{N_{n-2}} X(\chi) C_1(\cos \chi) \quad , \text{ and}$$

$$X_\mu(\theta_\mu) \propto \sin^{N_{n-\mu-2}} \theta_\mu C_\mu(\cos \theta_\mu) \quad (15)$$

with $\mu = 1, 2, 3, 4, \dots, n-4$.

We obtain

$$\begin{aligned} & (1-x^2) \frac{d^2}{dx^2} C(x) - x(2N_{n-2} + n - 1) \frac{d}{dx} C(x) \\ & + \left[\left(N^2 - \frac{(n-2)^2}{4} \right) - N_{n-2}(N_{n-2} + n - 2) \right] C(x) = 0 \quad , \end{aligned} \quad (16)$$

where $x = \cos \chi$.

$$\begin{aligned} & (1-x_\mu^2) \frac{d^2}{dx_\mu^2} C_\mu(x_\mu) - x_\mu(2N_{n-\mu-2} + n - \mu - 1) \frac{d}{dx_\mu} C_\mu(x_\mu) \\ & + [N_{n-\mu-1}(N_{n-\mu-1} + n - \mu - 2) - N_{n-\mu-2}(N_{n-\mu-2} + n - \mu - 2)] C_\mu(x_\mu) = 0, \end{aligned} \quad (17)$$

where $x_\mu = \cos \theta_\mu$, $\mu = 1, 2, 3, 4, \dots, n-4$.

The differential equation Eq. (17) can be solved by the method of Frobenius. All solution sets are also divergent at the end points of our interval. To obtain regular solutions in the

entire interval $x_\mu \in [-1, 1]$ ($\mu = 1, 2, 3, 4, \dots, n-4$) we terminate the infinite series after a finite number of terms by restricting N_3, N_4, \dots, N_{n-2} to have integer values.

$$\begin{aligned} N_{n-2} &= 0, 1, 2, 3, 4, \dots, \\ N_{n-\mu-1} &= 0, 1, 2, 3, 4, \dots, N_{n-\mu} \text{ with } \mu = 2, 3, 4, \dots, n-4, \\ (N_2 =) l &= 0, 1, 2, 3, 4, \dots, N_3 \\ m &= -l, -l+1, \dots, 0, 1, \dots, l. \end{aligned} \quad (18)$$

The polynomial solution obtained this way for the equation (17) is the Gegenbauer polynomials

$$X_\mu(\theta_\mu) \propto \sin^{N_{n-\mu-2}} \theta_\mu C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2}+\frac{n-\mu-2}{2}}(\cos \theta_\mu) \text{ with } \mu = 1, 2, 3, 4, \dots, n-4. \quad (19)$$

We return to Eq. (16). This differential equation can be solved by the method of Frobenius. Hence, we express the function $C(x)$ by a power series as

$$C(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha}. \quad (20)$$

This leads us to the following recursion relation:

$$\frac{a_{k+2}}{a_k} = \frac{\left[k + N_{n-2} + \frac{(n-2)^2}{2} \right] - N^2}{(k+1)(k+2)}. \quad (21)$$

The above recursion relation gives us the following series solution for $C(x)$:

$$C(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}. \quad (22)$$

Using the Recurrence relation this is in the form

$$C(x) = a_0 C_1(x) + a_1 C_2(x), \quad (23)$$

Where $C_1(x)$ and $C_2(x)$ are linearly independent. Convergence of the series at the end points of our interval can be checked easily by using the Raabe test which says that if $\sum_{k=0}^{\infty} u_k$ is a series of positive terms and if the $\lim_{n \rightarrow \infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) = A$ the series is divergent for

$A < 1$, convergent for $A > 1$ and the test fails for $A = 1$ [12]. At the end points we apply this test to the series with even powers i.e. $C_1(x)$. We write

$$\frac{a_{2k}}{a_{2k+1}} = \frac{(2k+1)(2k+2)}{\left[2k + N_{n-2} + \frac{(n-2)}{2}\right]^2 - N^2}, \quad (24)$$

one obtains $A = \frac{3}{2} - N_{n-2} - \frac{(n-2)}{2}$ (with $N_{n-2} \geq 0$ and $n \geq 4$) $A < 1$. Hence the infinite series $C_1(x)$ diverges at the end points. One may similarly check that $C_2(x)$ is also divergent at the end points of the interval. To obtain a regular solution in the entire interval $x \in [-1, 1]$ we terminate the infinite series after a finite number of terms by restricting N to have integer values given as

$$N(k, N_{n-2}) = k + N_{n-2} + \frac{(n-2)}{2}, \quad (25)$$

where $k = 0, 1, 2, 3, 4, \dots$ and $N_{n-2} = 0, 1, 2, 3, \dots$ with $\omega = \frac{N_{n-2}}{R_0}$. Redefining a new index k' as $k + N_{n-2}$ we write $N = k' + \frac{(n-2)}{2}$ with $\omega_k = \frac{N}{R_0}$ and $k' = 0, 1, 2, 3, \dots$, and $N_{n-2} = 0, 1, 2, 3, \dots, k'$. Dropping primes one obtains the eigenfrequencies as

$$\omega_k = \frac{1}{R_0} \left[k + \frac{(n-2)}{2} \right] \quad \text{with } k = 0, 1, 2, 3, 4, \dots \quad (26)$$

Values that $N(k, N_{n-2})$ could take are presented in table I..

For fixed k , one obtains the degeneracy of each eigenfrequencies

$$g_k = \sum_{N_{n-2}=0}^k \sum_{N_{n-3}=0}^{N_{n-2}} \dots \sum_{N_5=0}^{N_6} \sum_{N_4=0}^{N_5} \sum_{N_3=0}^{N_4} \sum_{l=0}^{N_3} (2l+1) \quad (27)$$

The polynomial solutions obtained this way are the well known Gegenbauer polynomials, hence we can write $C_1(x)$ in terms of the Gegenbauer polynomials as $C_1(x) = C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi)$.

Thus, the mode solution for $n \geq 4$ becomes

$$\Phi_\lambda(x) = c_{0n} e^{-i\omega_k t} \sin^{N_{n-2}} \chi C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi) \prod_{\mu=1}^{n-4} \left[\sin^{N_{n-\mu-2}} \theta_\mu C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2} + \frac{n-\mu-2}{2}}(\cos \theta_\mu) \right] Y_{lm}(\theta_{n-3}, \phi), \quad (28)$$

with

$$\begin{aligned}
\omega_k &= \frac{1}{R_0} \left[k + \frac{(n-2)}{2} \right] , \quad \text{where } k = 0, 1, 2, 3, 4, \dots , \\
N_{n-2} &= 0, 1, 2, 3, 4, \dots, k \\
N_{n-(\mu+1)} &= 0, 1, 2, 3, 4, \dots, N_{n-\mu} \quad \text{with } \mu = 2, 3, 4, \dots, n-4 , \\
(N_2 =) l &= 0, 1, 2, 3, 4, \dots, N_3 \quad \text{and} \\
m &= -l, -l+1, \dots, 0, 1, 2, \dots, l ,
\end{aligned} \tag{29}$$

and c_{0n} is the normalization constant.

To evaluate the normalization constant we write the scalar product

$$\begin{aligned}
(\Phi_{\lambda_1}, \Phi_{\lambda_2}) &= -i \int [\Phi_{\lambda_1}(x) \partial_t \Phi_{\lambda_2}^*(x) - \partial_t \Phi_{\lambda_1}(x) \Phi_{\lambda_2}^*(x)] \sqrt{-g} d^{n-1}x \\
&= \delta_{\lambda_1 \lambda_2} .
\end{aligned} \tag{30}$$

Φ_λ (λ stands for $k, N_{n-2}, N_{n-3}, \dots, N_4, N_3, (N_2 =)l$ and m values which are given in Eq. (29)) given Eq. (28) and, from a complete orthonormal set with respect to the product Eq. (30).

Using Eq. (30) we obtain c_{0n} as

$$\begin{aligned}
c_{0n} &= \frac{2^{N_{n-2} + \frac{(n-4)}{2}}}{R_0^{\frac{n-2}{2}}} \Gamma \left(N_{n-2} + \frac{(n-2)}{2} \right) \left[\frac{(k - N_{n-2})!}{\pi \Gamma(k + N_{n-2} + n - 2)} \right]^{\frac{1}{2}} \\
&\quad \prod_{\mu=1}^{n-4} \left\{ 2^{N_{n-\mu-2} + \frac{(n-\mu-3)}{2}} \Gamma \left(N_{n-\mu-2} + \frac{(n-\mu-2)}{2} \right) \right. \\
&\quad \left. \left[\frac{(N_{n-\mu-1} - N_{n-\mu-2})! \left(N_{n-\mu-1} + \frac{(n-\mu-2)}{2} \right)}{\pi \Gamma(N_{n-\mu-1} + N_{n-\mu-2} + n - \mu - 2)} \right]^{\frac{1}{2}} \right\} .
\end{aligned} \tag{31}$$

The general solution of the field equation can be written as sum over these modes in the form

$$\Phi(x) = \sum_{\lambda} \left(a_{\lambda} \Phi_{\lambda}(x) + a_{\lambda}^{\dagger} \Phi_{\lambda}^*(x) \right) . \tag{32}$$

Where, a_{λ} is an annihilation operator and a_{λ}^{\dagger} 's a creation operator for quanta in mode λ , while $\Phi_{\lambda}(x)$ is the solution of the wave equation given by Eq. (28).

III. THE GREEN'S FUNCTION

The Green's function we aim to calculate is defined by

$$\langle 0 | \Phi(x) \Phi(x') | 0 \rangle = \sum_{\lambda} \Phi_{\lambda}(x) \Phi_{\lambda}^*(x') , \quad (33)$$

where $\Phi_{\lambda}(x)$ are the mode function.

Using the mode solutions given in equation one obtains

$$\begin{aligned} G^{(+)}(x, x') = & \frac{1}{\pi R_0^{n-2}} \sum_{k=0}^{\infty} e^{-i \frac{\Delta t}{R_0} (k + \frac{(n-2)}{2})} \sum_{N_{n-2}=0}^k 2^{2N_{n-2}+n-4} \left\{ \Gamma \left(N_{n-2} + \frac{(n-2)}{2} \right) \right\}^2 \\ & \frac{(k - N_{n-2})!}{\Gamma(N_{n-2} + k + n - 2)} \sin^{N_{n-2}} \chi \sin^{N_{n-2}} \chi' C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi) C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi') \\ & \prod_{\mu=1}^{n-4} \left\{ \sum_{N_{n-\mu-2}=0}^{N_{n-\mu-1}} 2^{2N_{n-\mu-2}+n-\mu-3} \left\{ \Gamma \left(N_{n-\mu-2} + \frac{(n-\mu-2)}{2} \right) \right\}^2 \right. \\ & \left. \left[\frac{(N_{n-\mu-1} - N_{n-\mu-2})! \left(N_{n-\mu-1} + \frac{(n-\mu-2)}{2} \right)}{\pi \Gamma(N_{n-\mu-1} + N_{n-\mu-2} + n - \mu - 2)} \right] \right. \\ & \left. \sin^{N_{n-\mu-2}} \theta_{\mu} \sin^{N_{n-\mu-2}} \theta'_{\mu} C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2} + \frac{(n-\mu-2)}{2}}(\cos \theta_{\mu}) C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2} + \frac{(n-\mu-2)}{2}}(\cos \theta'_{\mu}) \right\} \\ & \sum_{m=-l}^l Y_{lm}(\theta_{n-3}, \phi) Y_{lm}^*(\theta'_{n-3}, \phi') . \end{aligned} \quad (34)$$

Using the addition formula for the spherical harmonics [12], which is given as

$$\begin{aligned} \frac{(2l+1)}{4\pi} P_l(\cos \gamma_1) = & \sum_{m=-l}^l Y_{lm}(\theta_{n-3}, \phi) Y_{lm}^*(\theta'_{n-3}, \phi') , \\ \text{where } \cos \gamma_1 = & \cos \theta_{n-3} \cos \theta'_{n-3} + \sin \theta_{n-3} \sin \theta'_{n-3} \cos(\phi - \phi') . \end{aligned} \quad (35)$$

And, using the $n-3$ times addition theorem for the Gegenbauer polynomials [13]

$$\begin{aligned} C_{\nu}^p(\cos \vartheta \cos \varphi + \sin \vartheta \sin \varphi \cos \psi) = & \frac{\Gamma(2p-1)}{\{\Gamma(p)\}^2} \sum_{\eta=0}^{\nu} 2^{2\eta} \{\Gamma(p+\eta)\}^2 \frac{(\nu-\eta)!(2\eta+2p-1)}{\Gamma(\nu+\eta+2p)} \\ & \sin^{\eta} \vartheta \sin^{\eta} \varphi C_{\nu-\eta}^{p+\eta}(\cos \vartheta) C_{\nu-\eta}^{p+\eta}(\cos \varphi) C_{\eta}^{p-\frac{1}{2}}(\cos \psi) . \end{aligned} \quad (36)$$

We obtain

$$G^{(+)}(x, x') = \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \sum_{k=0}^{\infty} e^{-i\frac{\Delta t}{R_0}(k+\frac{(n-2)}{2})} C_k^{\frac{(n-2)}{2}}(\cos \gamma_{n-2}) , \quad (37)$$

where

$$\alpha_n = 2^0.2^1.2^2....2^{n-4} \frac{\{\Gamma(1)\}^2 \{\Gamma(\frac{3}{2})\}^2 \{\Gamma(2)\}^2 \{\Gamma(\frac{n-2}{2})\}^2}{\Gamma(1)\Gamma(2)\Gamma(3)\Gamma(4).....\Gamma(n-3)} , \quad (38)$$

and

$$\begin{aligned} \cos \gamma_{n-2} &= \cos \chi \cos \chi' + \sin \chi \sin \chi' \cos \gamma_{n-3} , \\ \cos \gamma_{n-\mu-2} &= \cos \theta_\mu \cos \theta'_\mu + \sin \theta_\mu \sin \theta'_\mu \cos \gamma_{n-\mu-3} \\ \text{with } \mu &= 1, 2, 3, 4, \dots, n-4 , \text{ and} \end{aligned}$$

$$\cos \gamma_1 = \cos \theta_{n-3} \cos \theta'_{n-3} + \sin \theta_{n-3} \sin \theta'_{n-3} \cos(\phi - \phi') . \quad (39)$$

Substituting $\Delta t - i\epsilon$ for with the understanding that we will let $\epsilon \rightarrow 0$ at the end we get $\left| e^{-i\frac{\Delta t}{R_0}} \right| < 1$. Now, we could use the generating function for the Gegenbauer polynomials, one obtains

$$G^{(+)}(x, x') = \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \gamma_{n-2} \right)^{\frac{(n-2)}{2}}} . \quad (40)$$

Where γ_{n-2} is the angle between the vectors \mathbf{r} and \mathbf{r}' in the direction $(\chi, \theta_1, \theta_2, \theta_3, \dots, \theta_{n-3}, \phi)$ and $(\chi', \theta'_1, \theta'_2, \theta'_3, \dots, \theta'_{n-3}, \phi')$. The geodesic distance on S^{n-1} is denoted by $\Delta s_{n-2}(q, q') = R_0 \gamma_{n-2}$ where $q = q(\chi, \theta_1, \theta_2, \theta_3, \dots, \theta_{n-3}, \phi)$; q' and $q \in S^{n-1}$ and $x = x(t, q)$. The spacelike separation $\Delta s_{n-2}(q, q')$ is given in terms of the coordinates $(\chi, \theta_1, \theta_2, \theta_3, \dots, \theta_{n-3}, \phi)$ and $(\chi', \theta'_1, \theta'_2, \theta'_3, \dots, \theta'_{n-3}, \phi')$ as

$$(\cos \gamma_{n-2}) = \cos \frac{\Delta s_{n-2}}{R_0} = \cos \chi \cos \chi' + \sin \chi \sin \chi' \cos \gamma_{n-3} , \quad (41)$$

where $\cos \gamma_{n-3}$ is given in Eq. (39).

Thus, we have the following the two point function

$$\langle 0 | \Phi(x) \Phi(x') | 0 \rangle = \frac{\alpha_n}{4\pi^{n-2} R_0^{n-2}} \frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\Delta s_{n-2}}{R_0} \right)^{\frac{(n-2)}{2}}} . \quad (42)$$

Notice that the Hadamard Green's function $G^{(+)}(x, x')$ is related to $G^{(1)}(x, x')$ through the relation

$$\begin{aligned} G^{(1)}(x, x') &= 2\text{Re} \langle 0 | \Phi(x) \Phi(x') | 0 \rangle , \\ &= 2\text{Re} G^{(+)}(x, x') . \end{aligned} \quad (43)$$

IV. THE GREEN'S FUNCTION FOR HALF UNIVERSE

We will now investigate the massless conformal scalar field with a spherical boundary located at $\chi_0 = \frac{\pi}{2}$ and with interior geometry represented by the static closed metric. The metric inside the boundary may be expressed as

$$\begin{aligned} ds^2 &= dt^2 - R_0^2 [d\chi^2 + \sin^2 \chi d\theta_1^2 + \sin^2 \chi \sin^2 \theta_1 d\theta_2^2 + \dots \\ &\quad + \sin^2 \chi \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{n-4} d\theta_{n-3}^2 \\ &\quad + \sin^2 \chi \sin^2 \theta_1 \sin^2 \theta_2 \dots \sin^2 \theta_{n-4} \sin^2 \theta_{n-3} d\phi^2] , \end{aligned} \quad (44)$$

where $\chi \in [0, \frac{\pi}{2}]$, $\theta_\mu \in [0, \pi]$, $\mu = 1, 2, 3, 4, 5, \dots, n-3$ and $\phi \in [0, 2\pi]$.

We consider a massless conformal scalar field on this curved background geometry where the wave equation is given by

$$\square \Phi(x) + \xi(n) R \Phi(x) = 0. \quad (45)$$

Where $\xi(n) = \frac{1}{4} \frac{(n-2)}{(n-1)}$ is the conformal coupling constant and R is the Ricci scalar: $R = \frac{1}{4} (n-2)(n-1)$ for $n \geq 4$ (n is the dimension of spacetime).

Solution of Eq. (45) could be found easily

$$\Phi_\lambda(x) = \overline{c_{0n}} e^{-i\omega t} \sin^{N_{n-2}} \chi C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi) \prod_{\mu=1}^{n-4} \left[\sin^{N_{n-\mu-2}} \theta_\mu C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2} + \frac{(n-\mu-2)}{2}}(\cos \theta_\mu) \right] Y_{lm}(\theta_{n-3}, \phi) , \quad (46)$$

where $\overline{c_{0n}}$ is the normalization constant, and $\omega = \frac{N(k)}{R_0}$.

We note that the boundary condition at $\chi = \chi_0$ has not imposed on Eq. (46) yet. Hence, N still remains a continuous parameter, while $N_{n-2}, N_{n-(\mu+1)}$ ($\mu = 2, 3, 4, \dots, n-4$), l and m take the values.

$$\begin{aligned} N_{n-2} &= 0, 1, 2, 3, 4, 5, \dots \\ N_{n-(\mu+1)} &= 0, 1, 2, 3, 4, \dots, N_{n-\mu} , \quad \text{with } \mu = 2, 3, 4, \dots, n-4 , \\ (N_2)l &= 0, 1, 2, 3, 4, \dots, N_3 \quad \text{and} \\ m &= -l, -l+1, \dots, 0, 1, 2, \dots, l . \end{aligned} \quad (47)$$

We now impose the Dirichlet boundary condition for our problem i.e.

$$\Phi_\lambda(x)|_{\chi=\chi_0} = 0 \quad \text{at } \chi_0 = \frac{\pi}{2} . \quad (48)$$

Using the wave function given Eq. (46) we immediately get integer values for N , which could be tabulated as in table II.. The new eigenvalues $\omega_k = \frac{N(k)}{R_0}$ and the degeneracy g_k could be written as

$$\omega_k = \frac{1}{R_0} \left(k + \frac{(n-2)}{2} \right) , \quad k = 0, 1, 2, 3, 4, \dots$$

$$g_k = \left\{ \begin{array}{l} k = 1, N_{n-2} = 0 \\ k = 2, N_{n-2} = 1 \\ k = 3, N_{n-2} = 0, 2 \\ k = 4, N_{n-2} = 1, 3 \\ k = 5, N_{n-2} = 0, 2, 4 \\ \vdots \end{array} \right\} \cdot \sum_{N_{n-3}=0}^{N_{n-2}} \sum_{N_{n-4}=0}^{N_{n-3}} \dots \sum_{N_4=0}^{N_5} \sum_{N_3=0}^{N_4} \sum_{l=0}^{N_3} (2l+1) . \quad (49)$$

The new eigenfunctions Eq. (46) are still the Gegenbauer polynomials, but the only ones that satisfy the boundary condition $\Phi_\lambda(x)|_{\chi=\chi_0} = 0$ are picked: hence table II. could be formed by just taking the second, fourth, etc. diagonal elements of the table I..

Now we impose the Neumann boundary condition, which is given by

$$n^\alpha \nabla_\alpha \Phi_\lambda(x) = 0 \quad \text{at } \chi_0 = \frac{\pi}{2}, \quad (50)$$

where n^α is the outward normal to the boundary. Hence for the Neumann boundary condition

$$\frac{d}{d\theta} \Phi_\lambda(x) \big|_{\chi=\chi_0} = 0 \quad \text{at } \chi_0 = \frac{\pi}{2}. \quad (51)$$

We obtain the N values presented in table III. The new eigenvalues ω_k and the degeneracy could be written as

$$\omega_k = \frac{1}{R_0} \left(k + \frac{(n-2)}{2} \right), \quad k = 0, 1, 2, 3, 4, \dots$$

$$g_k = \left\{ \begin{array}{l} k = 0, N_{n-2} = 0 \\ k = 1, N_{n-2} = 1 \\ k = 2, N_{n-2} = 0, 2 \\ k = 3, N_{n-2} = 1, 3 \\ k = 4, N_{n-2} = 0, 2, 4 \\ \vdots \quad \quad \quad \vdots \end{array} \right\} \cdot \sum_{N_{n-3}=0}^{N_{n-2}} \sum_{N_{n-4}=0}^{N_{n-3}} \dots \sum_{N_4=0}^{N_5} \sum_{N_3=0}^{N_4} \sum_{l=0}^{N_3} (2l+1) \quad . \quad (52)$$

Notice that the N values in table III. could be obtained from the full universe case in table I. by taking the first, third, etc. diagonal elements. Once again, the eigenfunctions are the Gegenbauer polynomials, but only the ones that satisfy the Neumann boundary condition are allowed.

The Green's function that we aim to calculate is defined by

$$\langle 0 | \Phi(x) \Phi(x') | 0 \rangle = \sum_{\lambda} \Phi_{\lambda}(x) \Phi_{\lambda}^*(x'), \quad (53)$$

where $\Phi_{\lambda}(x)$ is the solution of the wave equation Eq. (45) with the appropriate boundary conditions Eqs. (48) and (51). Using the wave function given in Eq. (46), one obtains

$$\langle 0 | \Phi(x) \Phi(x') | 0 \rangle = \frac{2}{\pi R_0^{n-2}} \sum_{\lambda} e^{-i \frac{\Delta t}{R_0} \left(k + \frac{(n-2)}{2} \right)} 2^{2N_{n-2}+n-4} \left\{ \Gamma \left(N_{n-2} + \frac{(n-2)}{2} \right) \right\}^2$$

$$\begin{aligned}
& \frac{(k - N_{n-2})!}{\Gamma(N_{n-2} + k + n - 2)} \sin^{N_{n-2}} \chi \sin^{N_{n-2}} \chi' C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi) C_{k-N_{n-2}}^{N_{n-2} + \frac{(n-2)}{2}}(\cos \chi') \\
& \prod_{\mu=1}^{n-4} \left\{ \sum_{N_{n-\mu-2}=0}^{N_{n-\mu-1}} 2^{2N_{n-\mu-2} + n - \mu - 3} \left\{ \Gamma \left(N_{n-\mu-2} + \frac{(n-\mu-2)}{2} \right) \right\}^2 \right. \\
& \left[\frac{(N_{n-\mu-1} - N_{n-\mu-2})! \left(N_{n-\mu-1} + \frac{(n-\mu-2)}{2} \right)}{\pi \Gamma(N_{n-\mu-1} + N_{n-\mu-2} + n - \mu - 2)} \right] \\
& \left. \sin^{N_{n-\mu-2}} \theta_{\mu} \sin^{N_{n-\mu-2}} \theta'_{\mu} C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2} + \frac{(n-\mu-2)}{2}}(\cos \theta_{\mu}) C_{N_{n-\mu-1}-N_{n-\mu-2}}^{N_{n-\mu-2} + \frac{(n-\mu-2)}{2}}(\cos \theta'_{\mu}) \right\} \\
& \sum_{m=-l}^l Y_{lm}(\theta_{n-3}, \phi) Y_{lm}^*(\theta'_{n-3}, \phi') . \tag{54}
\end{aligned}$$

$\lambda(N_{n-2}, k)$ stands the new eigenvalues for the Dirichlet and Neumann boundary conditions are presented in tables II. and III..

We define the Green's function for the Dirichlet and Neumann boundary conditions as

$$\begin{aligned}
\langle 0 | \Phi_D(x) \Phi_D(x') | 0 \rangle &= D_D(x, x') , \text{ and} \\
\langle 0 | \Phi_N(x) \Phi_N(x') | 0 \rangle &= D_N(x, x') . \tag{55}
\end{aligned}$$

We easily see that

$$D_D(x, x') + D_N(x, x') = 2 D(x, x') . \tag{56}$$

Where $D(x, x') = \langle 0 | \Phi(x) \Phi(x') | 0 \rangle$ is the Green's function for the n-dimensional universe which given in Eq. (40). Since the sum of the Dirichlet and Neumann eigenvalues for the half universe which is depicted in tables II. and III. is equal to full n-dimensional universe eigenvalues given in table I.. The factor of 2 comes from the fact that the mode functions used in $D(x, x')$ are normalized with respect to the n-dimensional universe.

To evaluate $D_D(x, x')$ and $D_N(x, x')$ explicitly we find it convenient to write them as

$$D_D(x, x') = D(x, x') - \frac{1}{2} [D_N(x, x') - D_D(x, x')] , \text{ and} \tag{57}$$

$$D_N(x, x') = D(x, x') + \frac{1}{2} [D_N(x, x') - D_D(x, x')] . \tag{58}$$

The second term on the right hand side of Eqs. (57) and (58) could be written as $D_B(x, x') = \frac{1}{2} [D_N(x, x') - D_D(x, x')] .$ A closed expression for the $D(x, x')$ is already known which given in Eq. (40) and could be written as

$$(G^{(+)}(x, x') =) D(x, x') = \frac{\alpha_n}{4\pi^{n-2} R_0^{n-2}} \frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\Delta s_{n-2}}{R_0} \right)^{\frac{(n-2)}{2}}} . \quad (59)$$

Where $\Delta t = t - t'$, and

$$\cos \frac{\Delta s_{n-2}}{R_0} = \cos \chi \cos \chi' + \sin \chi \sin \chi' \cos \gamma_{n-3} . \quad (60)$$

Here $\cos \gamma_{n-3}$ and α_n are given by Eq. (38) and Eq. (39), respectively.

Using the information given in tables II. and III., definitions of $D_N(x, x')$ and $D_D(x, x')$, and addition theorem for the Gegenbauer polynomials one could write as

$$\begin{aligned} D_B(x, x') = & \frac{\alpha_n}{4\pi^{n-2} R_0^{n-2}} \frac{\Gamma(n-3)}{\left\{ \Gamma\left(\frac{n-2}{2}\right) \right\}^2} \left[e^{-i \frac{(n-2)\Delta t}{2R_0}} \frac{\left\{ \Gamma\left(\frac{n-2}{2}\right) \right\}^2}{\Gamma(n-2)} (n-3) \right. \\ & C_0^{\frac{(n-2)}{2}}(\cos \chi) C_0^{\frac{(n-2)}{2}}(\cos \chi') C_0^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) \\ & + e^{-i \frac{n\Delta t}{2R_0}} \left\{ \frac{2 \left\{ \Gamma\left(\frac{n}{2}\right) \right\}^2}{\Gamma(n)} (n-1) \sin \chi \sin \chi' \right. \\ & C_0^{\frac{n}{2}}(\cos \chi) C_0^{\frac{n}{2}}(\cos \chi') C_1^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) \\ & \left. \left. - \frac{\left\{ \Gamma\left(\frac{n-2}{2}\right) \right\}^2}{\Gamma(n-1)} (n-3) C_1^{\frac{(n-2)}{2}}(\cos \chi) C_1^{\frac{(n-2)}{2}}(\cos \chi') C_0^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) \right\} \right. \\ & + e^{-i \frac{(n+2)\Delta t}{2R_0}} \left\{ \frac{2! \left\{ \Gamma\left(\frac{n-2}{2}\right) \right\}^2}{\Gamma(n)} (n-3) C_2^{\frac{(n-2)}{2}}(\cos \chi) \right. \\ & C_2^{\frac{(n-2)}{2}}(\cos \chi') C_0^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) \\ & \left. - \frac{2^2 \left\{ \Gamma\left(\frac{n}{2}\right) \right\}^2 1!}{\Gamma(n+1)} (n-1) \sin \chi \sin \chi' \right. \\ & C_1^{\frac{n}{2}}(\cos \chi) C_1^{\frac{n}{2}}(\cos \chi') C_1^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) \\ & + \frac{2^4 \left\{ \Gamma\left(\frac{n+6}{2}\right) \right\}^2 0!}{\Gamma(n+2)} (n+1) \sin^2 \chi \sin^2 \chi' C_0^{\frac{(n+2)}{2}}(\cos \chi) \\ & \left. \left. C_0^{\frac{(n+2)}{2}}(\cos \chi') C_2^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) \right\} + \dots \right] . \quad (61) \end{aligned}$$

Using the symmetry properties of Gegenbauer polynomials

$$C_{k-N_{n-2}}^{N_{n-2}+\frac{(n-2)}{2}}(-\cos \chi') = (-1)^{k-N_{n-2}} C_{k-N_{n-2}}^{N_{n-2}+\frac{(n-2)}{2}}(\cos \chi') , \quad (62)$$

we could now rewrite Eq. (61) as

$$\begin{aligned} D_B(x, x') &= \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \frac{\Gamma(n-3)}{\left\{\Gamma\left(\frac{n-2}{2}\right)\right\}^2} \sum_{k=0}^{\infty} e^{-i\omega_k \Delta t} \sum_{N_{n-2}=0}^k 2^{N_{n-2}} \frac{\left\{\Gamma\left(N_{n-2} + \frac{(n-2)}{2}\right)\right\}^2}{\Gamma(N_{n-2} + k + (n-2))} \\ &\quad (2N_{n-2} + n - 3) \sin^{N_{n-2}} \chi \sin^{N_{n-2}} \chi' \\ &\quad C_{k-N_{n-2}}^{N_{n-2}+\frac{(n-2)}{2}}(\cos \chi) C_{k-N_{n-2}}^{N_{n-2}+\frac{(n-2)}{2}}(-\cos \chi) C_{N_{n-2}}^{\frac{(n-3)}{2}}(\cos \gamma_{n-3}) . \end{aligned} \quad (63)$$

Using the addition theorem for the Gegenbauer polynomials, one obtains

$$D_B(x, x') = \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \sum_{k=0}^{\infty} e^{-i\omega_k \Delta t} C_k^{\frac{(n-2)}{2}}(\cos \widetilde{\gamma_{n-2}}) , \quad (64)$$

where

$$\cos \widetilde{\gamma_{n-2}} = -\cos \chi \cos \chi' + \sin \chi \sin \chi' \cos \gamma_{n-3} . \quad (65)$$

Letting $\Delta t \rightarrow \Delta t - i\epsilon$ where $\epsilon \rightarrow 0$, then $\left|e^{-i\frac{\Delta t}{R_0}}\right| < 1$. Now the sum in Eq. (64) could be evaluated using the generating function of Gegenbauer polynomials, one obtains

$$D_B(x, x') = \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\widetilde{\Delta s_{n-2}}}{R_0}\right)^{\frac{(n-2)}{2}}} , \quad (66)$$

where

$$\begin{aligned} (\cos \widetilde{\gamma_{n-2}}) &= \cos \frac{\widetilde{\Delta s_{n-2}}}{R_0} = -\cos \chi \cos \chi' + \sin \chi \sin \chi' \cos \gamma_{n-3} , \\ \cos \gamma_{n-\mu-2} &= \cos \theta_\mu \cos \theta'_\mu + \sin \theta_\mu \sin \theta'_\mu \cos \gamma_{n-\mu-3} \\ &\text{with } \mu = 2, 3, 4, \dots, n-4 , \text{ and} \\ \cos \gamma_1 &= \cos \theta_{n-3} \cos \theta'_{n-3} + \sin \theta_{n-3} \sin \theta'_{n-3} \cos(\phi - \phi') . \end{aligned} \quad (67)$$

We could now write the complete Green's functions $D_N(x, x')$ and $D_D(x, x')$ as

$$D_N(x, x') = \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \left[\frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\Delta s_{n-2}}{R_0}\right)^{\frac{(n-2)}{2}}} + \frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\widetilde{\Delta s_{n-2}}}{R_0}\right)^{\frac{(n-2)}{2}}} \right] , \quad (68)$$

$$D_D(x, x') = \frac{\alpha_n}{4\pi^{n-2}R_0^{n-2}} \left[\frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\Delta s_{n-2}}{R_0}\right)^{\frac{(n-2)}{2}}} - \frac{1}{\left(\cos \frac{\Delta t}{R_0} - \cos \frac{\widetilde{\Delta s_{n-2}}}{R_0}\right)^{\frac{(n-2)}{2}}} \right] . \quad (69)$$

Where $\cos \frac{\Delta s_{n-2}}{R_0}$ and $\cos \frac{\widetilde{\Delta s_{n-2}}}{R_0}$ are given by Eq. (39) and Eq. (67), respectively.

We also note that these Green's functions are naturally identical to the Green's functions that one could get by using the image method, which always works on the double manifold defined by [11,14,15]

$$M \cup \partial M \cup M^* . \quad (70)$$

M is the physical space inside the boundary, ∂M is the boundary and M^* is the dual space obtained by reflecting the physical space about the boundary. The field is confined to the region defined by M and satisfies the boundary conditions Eqs. (48) and (50). To satisfy the boundary condition on ∂M it is sufficient to locate an image charge in the unphysical dual region M^* . The Green's function on $M \cup \partial M$ is then given by

$$G(x, x') = D(x, x') \pm D(x, \tilde{x}') , \quad (71)$$

where $- (+)$ refers to the Dirichlet (Neumann) boundary condition, $D(x, x')$ is the Green's function for the double manifold and \tilde{x}' is the image of x' . The key element in the success of image method is that the solutions of Eq. (45) found in the double manifold will have distinct parity (even or odd) under transformation $x \rightarrow \tilde{x}$. Notice that in Eq. (68) and Eq. (69) the first term is the Green's function for the double manifold, which in this case is the full S^{n-1} , and the $(\cos \widetilde{\gamma_{n-2}} =) \cos \frac{\widetilde{\Delta s_{n-2}}}{R_0}$ in the second term could be rewritten as

$$(\cos \widetilde{\gamma_{n-2}} =) \cos \frac{\widetilde{\Delta s_{n-2}}}{R_0} = -\cos \chi \cos \chi' + \sin \chi \sin \chi' \cos \gamma_{n-3} , \quad (72)$$

which could be obtained from $\cos \frac{\Delta s_{n-2}}{R_0}$ by substituting the image of x' i.e. by replacing $\chi' \rightarrow \pi - \chi'$.

V. SUMMARY

We constructed the Green's function by using the eigenfunctions, which are obtained by solving the wave equation for a conformal scalar field in a n-dimensional closed, static universe and with the appropriate boundary conditions (Dirichlet and Neumann). Even though

we use the global topology of the n -dimensional universe to construct the Green's function, our result is locally true. Our result is interesting, to understand the correspondence between the sign of the Casimir energy and different manifold structure in closed topology of the universe.

Our result is in agreement with the results obtained by the method of images. The Green's function is calculated for an isolated sphere, where the interior geometry is given by Eq. (44) (half space universe). The global topology of the universe as well as the outside geometry are not important and have no effect on our result. This is very important for applications of the Casimir effect to the bag models. Currently we are investigating the Casimir effect for a massless conformal scalar field in a n -dimensional closed, static universe and a half space with Dirichlet and Neumann boundary conditions.

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Table I. : $N(k, N_{n-2})$ values for the full space case

$N_{n-2} \setminus k$	0	1	2	3	4	5	\dots
0	$\frac{n-2}{2}$	$\frac{n}{2}$	$\frac{n+2}{2}$	$\frac{n+4}{2}$	$\frac{n+6}{2}$	$\frac{n+8}{2}$	\dots
1		$\frac{n}{2}$	$\frac{n+2}{2}$	$\frac{n+4}{2}$	$\frac{n+6}{2}$	$\frac{n+8}{2}$	\dots
2			$\frac{n+2}{2}$	$\frac{n+4}{2}$	$\frac{n+6}{2}$	$\frac{n+8}{2}$	\dots
3				$\frac{n+4}{2}$	$\frac{n+6}{2}$	$\frac{n+8}{2}$	\dots
4					$\frac{n+6}{2}$	$\frac{n+8}{2}$	\dots
5						$\frac{n+8}{2}$	\dots
\vdots							\ddots

Table II.: $N(k, N_{n-2})$ values for the Dirichlet boundary case

$N_{n-2} \setminus k$	0	1	2	3	4	5	\dots
0		$\frac{n}{2}$		$\frac{n+4}{2}$		$\frac{n+8}{2}$	
1			$\frac{n+2}{2}$		$\frac{n+6}{2}$		\ddots
2				$\frac{n+4}{2}$		$\frac{n+8}{2}$	
3					$\frac{n+6}{2}$		\ddots
4						$\frac{n+8}{2}$	
5							\ddots
\vdots							

Table III.: $N(k, N_{n-2})$ values for the Neumann boundary case.

$N_{n-2} \setminus k$	0	1	2	3	4	5	\dots
0	$\frac{n-2}{2}$		$\frac{n+2}{2}$		$\frac{n+6}{2}$		\ddots
1		$\frac{n}{2}$		$\frac{n+4}{2}$		$\frac{n+8}{2}$	
2			$\frac{n+2}{2}$		$\frac{n+6}{2}$		\ddots
3				$\frac{n+4}{2}$		$\frac{n+8}{2}$	
4					$\frac{n+6}{2}$		\ddots
5						$\frac{n+8}{2}$	
\vdots							\ddots